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SMALL ZEROS OF QUADRATIC FORMS OVER NUMBER FIELDS. II

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ABSTRACT. Let F be a nontrivial quadratic form in N variables with coefficients in a number field k and let $\mathscr{Z}$ be a subspace of k^N of dimension M, $1 \le M \le N$. If F restricted to $\mathscr{Z}$ vanishes on a subspace of dimension L, $1 \le L < M$, and if the rank of F restricted to $\mathscr{Z}$ is greater than $M - L$, then we show that F must vanish on $M - L + 1$ distinct subspaces $\mathscr{X}_0, \mathscr{X}_1, \dots, \mathscr{X}_{M-L}$ in $\mathscr{Z}$ each of which has dimension L. Moreover, we show that for each pair $\mathscr{X}_0, \mathscr{X}_l$, $1 \le l \le M - L$, the product of their heights $H(\mathscr{X}_0)H(\mathscr{X}_l)$ is relatively small. Our results generalize recent work of Schlickewei and Schmidt.

1. INTRODUCTION

Let

$$F(\mathbf{x}, \mathbf{y}) = \sum_{m=1}^{N} \sum_{n=1}^{N} \varphi_{mn} y_m x_n \tag{1.1}$$

be a symmetric bilinear form with coefficients $\varphi_{mn} = \varphi_{nm}$ in an algebraic number field k. We write $\Phi = (\varphi_{mn})$ for the associated $N \times N$ matrix and $F(\mathbf{x}) = F(\mathbf{x}, \mathbf{x})$ for the associated quadratic form. As in our earlier paper [14] we will consider F restricted to a fixed subspace $\mathscr{Z} \subseteq k^N$ of dimension M, $1 \le M \le N$, and define

$$\mathscr{Z}^{(0)} = \{\mathbf{z} \in \mathscr{Z} : F(\mathbf{z}) = 0\}.$$

A basic problem in this situation is to show that if $\mathscr{Z}^{(0)}$ is not trivial then it necessarily contains vectors or subspaces of small height. Beginning with a result of Cassels [3, 4], the papers [1, 5–10, 12–15], are all directed at this type of problem. In case $k = \mathbb{Q}$ and $M = N$ Schlickewei and Schmidt [10] have recently proved a theorem which includes most of the previous results. Our purpose in the present paper is to generalize the work of Schlickewei and Schmidt to an arbitrary number field k and to the case where $\mathscr{Z}$ may be a proper subspace of k^N. As we have already noted in [14], the introduction of the subspace $\mathscr{Z}$ is equivalent to considering the simultaneous zeros of the

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quadratic form F and a system of $N-M$ independent linear forms. If A is an $(N-M)\times N$ matrix over k with $\operatorname{rank}(A)=N-M$ and

$$\mathscr{Z}=\{\mathbf{x}\in k^N : A\mathbf{x}=\mathbf{0}\},$$

then by a basic duality principle (equation (2.6) of [14]) the subspace $\mathscr{Z}$ and the matrix A have the same height. For this reason all of our results which we state for a quadratic form F restricted to $\mathscr{Z}$ have obvious analogs for the set of simultaneous solutions in k^N of $F(\mathbf{x})=0$ and $A\mathbf{x}=\mathbf{0}$.

We suppose that the number field k has degree d over $\mathbb{Q}$. Our notation for places, completions of k, normalized absolute values, heights and measures will be identical to that which was already described in [14, §2]. Also, we will assume that F restricted to $\mathscr{Z}$ has rank r. Our main result which generalizes Theorem 1 in [10] is as follows.

Theorem 1. *Suppose that $\mathscr{Z}^{(0)}$ contains an L-dimensional subspace, $1\le L<M$, and F restricted to $\mathscr{Z}$ has rank $r>M-L$. Then there exist $M-L+1$ distinct L-dimensional subspaces $\mathscr{X}_0, \mathscr{X}_1, \mathscr{X}_2, \dots, \mathscr{X}_{M-L}$ in $\mathscr{Z}^{(0)}$ with the following properties*:

(i) *for each $l=1,2,\dots,M-L$ the subspace $\mathscr{X}_0\cap\mathscr{X}_l$ has dimension $L-1$*;
(ii) *the union $\mathscr{X}_0\cup\mathscr{X}_1\cup\cdots\cup\mathscr{X}_{M-L}$ spans the subspace $\mathscr{Z}$*;
(iii) *for each $l=1,2,\dots,M-L$,*

$$H(\mathscr{X}_0)^2\le H(\mathscr{X}_0)H(\mathscr{X}_l)\le\{2c_k(M-L)^2\mathscr{H}(\Phi)\}^{M-L}\{2c_k(1)H(\mathscr{Z})\}^2. \tag{1.2}$$

(For each positive integer n, $c_k(n)$ is a field constant defined in [14, §2] and in (2.2) below.)

In our previous result [14, Theorem 1] we made no assumption concerning the rank of F restricted to $\mathscr{Z}$ but instead we assumed that the integer L was maximal. That is, we assumed that L was the largest integer such that $\mathscr{Z}^{(0)}$ contains a subspace of dimension L. We note that in the present paper we are *not* assuming that L is maximal. Of course $\mathscr{Z}^{(0)}$ must contain the subspace

$$\mathscr{Z}^{\perp}=\{\mathbf{z}\in\mathscr{Z} : F(\boldsymbol{\zeta},\mathbf{z})=0 \text{ for all } \boldsymbol{\zeta}\in\mathscr{Z}\},$$

and $\dim(\mathscr{Z}^{\perp})=M-r$. Thus our hypothesis in Theorem 1 concerning the rank of F restricted to $\mathscr{Z}$ could be stated this way: we assume that $\mathscr{Z}^{(0)}$ contains a subspace of dimension L with $L>\dim(\mathscr{Z}^{\perp})$. It turns out that the method used to prove Theorem 1 also provides a bound on the height of the subspace $\mathscr{Z}^{\perp}$.

Theorem 2. *If $1\le r<M$ then*

$$H(\mathscr{Z}^{\perp})\le c_k(r)^r\mathscr{H}(\Phi)^{r/2}H(\mathscr{Z}). \tag{1.3}$$

Suppose that in Theorem 1 the quadratic form F restricted to $\mathscr{Z}$ has rank $r=M$. Then we may take $L=1$. We find that if F has a nontrivial zero in $\mathscr{Z}$, then there is a basis $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{M-1}\}$ of $\mathscr{Z}$ such that $F(\mathbf{x}_m) = 0$, $m = 0, 1, \ldots, M-1$, and

$$H(\mathbf{x}_0)H(\mathbf{x}_l) \leq \{2c_k(M-1)^2\mathscr{H}(\Phi)\}^{M-1}\{2c_k(1)H(\mathscr{Z})\}^2, \tag{1.4}$$

for $l = 1, 2, \ldots, M-1$. Obviously (1.4) implies that

$$\begin{aligned} &H(\mathbf{x}_0)^{M-1}H(\mathbf{x}_1)H(\mathbf{x}_2)\cdots H(\mathbf{x}_{M-1}) \\ &\quad \leq \{2c_k(M-1)^2\mathscr{H}(\Phi)\}^{(M-1)^2}\{2c_k(1)H(\mathscr{Z})\}^{2(M-1)}. \end{aligned} \tag{1.5}$$

The bounds (1.4) and (1.5) generalize results of Schulze-Pillot [13, Theorem 2] and Chalk [5].

If F and $\mathscr{Z}$ satisfy the hypotheses in Theorem 1 then $\mathscr{Z}^{(0)}$ contains the L-dimensional subspace $\mathscr{X}_0$, and the height $H(\mathscr{X}_0)$ is bounded by (1.2). In fact the slightly sharper bound

$$H(\mathscr{X}_0) \leq \{2c_k(M-L)^2\mathscr{H}(\Phi)\}^{(M-L)/2}H(\mathscr{Z}) \tag{1.6}$$

follows immediately from Theorem 4 and equation (5.11) below. Theorem 2 and (1.6) provide a sharper and more general formulation of our previous result [14, Theorem 1].

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2. Preliminary lemmas

If v is a place of k we write k_v for the completion of k at v. Let $\mathscr{A} \subseteq (k)^N$ be a subspace of dimension L, $1 \leq L < N$, and let $\mathscr{A}_v \subseteq (k_v)^N$ be the completion of $\mathscr{A}$ in $(k_v)^N$. We shall make frequent use of the $N \times N$ projection matrices $P_v = P_v(\mathscr{A}_v)$ which were defined in [14, §4]. Here we simply summarize the main results concerning $P_v(\mathscr{A}_v)$ which we will need. Complete proofs and further details can be found in [14].

At each place v the matrix P_v is a projection operator in the usual sense:

$$P_v\mathbf{x} \in \mathscr{A}_v \quad \text{for all } \mathbf{x} \in (k_v)^N,$$

and

$$P_v\mathbf{x} = \mathbf{x} \quad \text{for all } \mathbf{x} \in \mathscr{A}_v.$$

Let $\mathbf{b} \in (k)^N \backslash \mathscr{A}$ so that

$$\mathscr{B} = \operatorname*{span}_k\{\mathscr{A}, \mathbf{b}\}$$

is a subspace of dimension $L+1$. The height of the subspaces $\mathscr{A}$ and $\mathscr{B}$ are related by the identity

$$H(\mathscr{B}) = H(\mathscr{A})\prod_v H_v\{(1_N - P_v)\mathbf{b}\}. \tag{2.1}$$

This follows from [14, Lemma 4].

If $\mathscr{A} \subseteq k^N$ we shall sometimes simplify our notation and write $P_v(\mathscr{A})$ for *projection onto the completion of* $\mathscr{A}$ *in* $(k_v)^N$.

Next we require a lemma whose proof uses methods from geometry of numbers over adèle spaces. The relevant definitions concerning this subject are contained in [2, pp. 16–18]. We write $k_{\mathbf{A}}$ for the adèle ring of the number field k and if v is a finite place of k then

$$O_v = \{x \in k_v : |x|_v \leq 1\}$$

denotes the maximal compact subring of k_v. Since the additive group of k_v is locally compact we may select a normalized Haar measure β_v on k_v as follows:

(i) If $v \mid p$, where p is a (finite) rational prime, we require that $\beta_v(O_v) = |\mathscr{D}_v|_v^{d/2}$. Here $\mathscr{D}_v$ is the local different of k at v.

(ii) If $k_v = \mathbb{R}$ then β_v is ordinary Lebesgue measure on $\mathbb{R}$.

(iii) If $k_v = \mathbb{C}$ then β_v is Lebesgue measure on the complex plane multiplied by 2.

The product measure $\beta = \prod_v \beta_v$ then induces a normalized Haar measure (also denoted by β) on $k_{\mathbf{A}}$. If $(k_{\mathbf{A}})^N$ is an N-fold product of adèle spaces we write V for the product Haar measure β^N on $(k_{\mathbf{A}})^N$.

At each infinite place v we define a positive real number $r_v(N)$ so that

$$\beta_v^N(\{\mathbf{u} \in (k_v)^N : \|\mathbf{u}\|_v < r_v(N)\}) = 1.$$

The exact value of $r_v(N)$ is given in [14, §2]. We also define

$$c_k(N) = \left\{ 2|\Delta_k|^{1/2d} \prod_{v|\infty} (r_v(N))^{d_v/d} \right\}, \tag{2.2}$$

where Δ_k is the discriminant of k and $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. The quantity $c_k(N)$ will occur as a field constant.

Now suppose that A_v is an $N \times M$ matrix over k_v with

$$\operatorname{rank}(A_v) = M \leq N.$$

Let $\boldsymbol{\xi}_v \neq \mathbf{0}$ be a vector in $(k_v)^N$. If $v \mid \infty$ we set

$$S_v = \{\mathbf{u} \in (k_v)^M : \|A_v\mathbf{u}\|_v < \|\boldsymbol{\xi}_v\|_v\}.$$

It follows easily that

$$\beta_v^M(S_v) = r_v(M)^{-Md_v}\|\boldsymbol{\xi}\|_v^{Md_v} H_v(A_v)^{-d}.$$

If $v \nmid \infty$ we write

$$S_v = \{\mathbf{u} \in (k_v)^M : \|A_v\mathbf{u}\|_v \leq \|\boldsymbol{\xi}_v\|_v\}.$$

By using the v-adic cube slicing identity, which is (4.8) and (4.9) of [2], the Haar measure of S_v is given by

$$\beta_v^M(S_v) = |\mathscr{D}_v|_v^{Md/2}\|\boldsymbol{\xi}_v\|_v^{Md_v} H_v(A_v)^{-d}.$$

If we assume that $S_v = (O_v)^M$ at almost all finite places v, then the set

$$\mathscr{S} = \prod_v S_v$$

is a subset of the M-fold product $(k_{\mathbf{A}})^M$. Hence the Haar measure of $\mathscr{S}$ is given by
(2.3)

$$V(\mathscr{S}) = \prod_v \beta_v^M(S_v) = |\Delta_k|^{-M/2}\left\{\prod_v |\boldsymbol{\xi}_v|_v^M H_v(A_v)^{-1}\right\}^d \left\{\prod_{v|\infty} r_v(M)^{-Md_v}\right\}.$$

(In (2.3) we have used the identity

$$\prod_{v\nmid\infty} |\mathscr{D}_v|_v^{d/2} = |\Delta_k|^{-1/2}.)$$

If $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_M < \infty$ denote the successive minima of $\mathscr{S}$ then by the adèlic form of Minkowski's second theorem (this is Theorem 3 of [2]) we have

$$(\lambda_1\lambda_2\cdots\lambda_M)^d V(\mathscr{S}) \le 2^{dM}.$$

Using (2.3) this may be written as

$$(\lambda_1\lambda_2\cdots\lambda_M) \le c_k(M)^M \left(\prod_v |\boldsymbol{\xi}_v|_v^{-M} H_v(A_v)\right). \tag{2.4}$$

To simplify the statement of the following lemma we write

$$G(\mathbf{x}) = \prod_v H_v\{(1_N - P_v)\mathbf{x}\}$$

for $\mathbf{x} \in k$ and $P_v = P_v(\mathscr{A}_v)$.

Lemma 3. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ be linearly independent vectors in $(k)^N \backslash \mathscr{A}$. Then there exists a scalar $\alpha \ne 0$ in k such that*

$$G(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi}) \le 2c_k(1)\max\{G(\boldsymbol{\zeta}), G(\boldsymbol{\xi})\}.$$

Proof. We use the inequality (2.4) with $M = 1$, $A_v = (1_N - P_v)\boldsymbol{\zeta}$, and $\boldsymbol{\xi}_v = (1_N - P_v)\boldsymbol{\xi}$. For almost all finite places we have

$$\|A_v\|_v = \|\boldsymbol{\xi}_v\|_v = 1$$

and therefore $S_v = (O_v)^M$ at almost all finite places. Hence the inequality (2.4) holds. It follows that there exists $\alpha \ne 0$ in k such that $\alpha \in \lambda\mathscr{S}$ for all $\lambda > \lambda_1$. That is,

$$\|(1_N - P_v)\alpha\boldsymbol{\zeta}\|_v \le \lambda_1 \|(1_N - P_v)\boldsymbol{\xi}\|_v$$

if $v \mid \infty$ and

$$\|(1_N - P_v)\alpha\boldsymbol{\xi}\|_v \le \|(1_N - P_v)\boldsymbol{\xi}\|_v$$

if $v \nmid \infty$. Thus we have

$$\prod_v |(1_N - P_v)(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi})|_v \le \left\{ \prod_{v|\infty} (1+\lambda_1)^{d_v/d} \right\} \prod_v |(1_N - P_v)\boldsymbol{\xi}|_v ,$$

or

$$G(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi}) \le (1+\lambda_1) G(\boldsymbol{\xi}). \tag{2.5}$$

Of course (2.4) may be written as

$$\lambda_1 \le c_k(1) G(\boldsymbol{\xi})^{-1} G(\boldsymbol{\zeta})$$

and therefore (2.5) implies that

$$G(\alpha\boldsymbol{\zeta} + \boldsymbol{\xi}) \le G(\boldsymbol{\xi}) + c_k(1) G(\boldsymbol{\zeta}) \le 2c_k(1) \max\{G(\boldsymbol{\zeta}), G(\boldsymbol{\xi})\}.$$

This proves the lemma.

Let $F(\mathbf{x}, \mathbf{y})$ be the bilinear form defined in (1.1) by the $N \times N$ symmetric matrix $\Phi = (\varphi_{mn})$. At each place v of k the local height $\mathscr{H}_v$ is defined by

$$\mathscr{H}_v(\Phi) = \max_{1 \le m, n \le N} |\varphi_{mn}|_v \quad \text{if } v \nmid \infty,$$

$$\mathscr{H}_v(\Phi) = \left\{ \sum_{m=1}^{N} \sum_{n=1}^{N} \|\varphi_{mn}\|_v^2 \right\}^{d_v/2d} \quad \text{if } v \mid \infty.$$

The global height of Φ is then given by

$$\mathscr{H}(\Phi) = \prod_v \mathscr{H}_v(\Phi).$$

If $\boldsymbol{\xi}_v$ and $\boldsymbol{\zeta}_v$ are vectors in $(k_v)^N$ we have

$$|F(\boldsymbol{\xi}_v, \boldsymbol{\zeta}_v)|_v \le \mathscr{H}_v(\Phi) |\boldsymbol{\xi}_v|_v |\boldsymbol{\zeta}_v|_v. \tag{2.6}$$

This follows immediately from the ultrametric inequality if $v \nmid \infty$ and from the Cauchy-Schwarz inequality if $v \mid \infty$.

Now suppose that $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ are vectors in k^N, F vanishes identically on $\mathscr{A} \subseteq \mathscr{Z} \subseteq k^N$ and $\boldsymbol{\zeta} \in \mathscr{A}^{\perp}$, where

$$\mathscr{A}^{\perp} = \{\mathbf{z} \in \mathscr{Z} : F(\mathbf{a}, \mathbf{z}) = 0 \text{ for all } \mathbf{a} \in \mathscr{A}\}.$$

By continuity F vanishes identically on the completion $\mathscr{A}_v$ of $\mathscr{A}$ in $(k_v)^N$. It follows that $F(P_v\boldsymbol{\xi}, \boldsymbol{\zeta}) = 0$ and therefore

$$F(\boldsymbol{\xi}, \boldsymbol{\zeta}) = F((1_N - P_v)\boldsymbol{\xi}, \boldsymbol{\zeta}). \tag{2.7}$$

If $F(\boldsymbol{\xi}, \boldsymbol{\zeta}) \neq 0$, then (2.6), (2.7) and the product formula imply that

$$\begin{aligned} 1 &= \prod_v |F(\boldsymbol{\xi}, \boldsymbol{\zeta})|_v \\ &= \prod_v |F((1_N - P_v)\boldsymbol{\xi}, \boldsymbol{\zeta})|_v \\ &\leq \prod_v \{\mathscr{H}_v(\Phi) H_v((1_N - P_v)\boldsymbol{\xi}) H_v(\boldsymbol{\zeta})\} \\ &= \mathscr{H}(\Phi) \left\{ \prod_v H_v((1_N - P_v)\boldsymbol{\xi}) \right\} H(\boldsymbol{\zeta}). \end{aligned}$$

This type of argument will be used frequently.

3. Two basic theorems

Our proof of Theorem 1 divides naturally into two parts and each part can be easily formulated as a separate result. We assume throughout §§3–6 of this paper that the hypotheses in Theorem 1 hold. Then among all L-dimensional subspaces contained in $\mathscr{Z}^{(0)}$ let $\mathscr{X}$ be an L-dimensional subspace with minimal height. As the set of L-dimensional subspaces of $\mathscr{Z}$ having height less than a positive constant is a finite set, such a subspace $\mathscr{X}$ clearly exists. We define

$$\mathscr{X}^{\perp} = \{\mathbf{z} \in \mathscr{Z} : F(\mathbf{x}, \mathbf{z}) = 0 \text{ for all } \mathbf{x} \in \mathscr{X}\}.$$

Since F restricted to $\mathscr{Z}$ has rank $r > M - L$, it follows that $\mathscr{X}^{\perp}$ is a proper subspace of $\mathscr{Z}$.

Theorem 4. *Suppose that $\boldsymbol{\xi}$ is a vector in $\mathscr{Z} \backslash \mathscr{X}^{\perp}$ and let $\mathscr{Y}$ be the $(L+1)$-dimensional subspace*

$$\mathscr{Y} = \operatorname{span}_k\{\mathscr{X}, \boldsymbol{\xi}\}.$$

Then there exists a subspace $\mathscr{X}' \subseteq \mathscr{Y}$ such that

(i) *F vanishes identically on $\mathscr{X}'$,*

(ii) $\dim(\mathscr{X}') = L$ *and* $\dim(\mathscr{X} \cap \mathscr{X}') = L - 1$,

(iii) $H(\mathscr{X})^2 \leq H(\mathscr{X}) H(\mathscr{X}') \leq 2\mathscr{H}(\Phi) H(\mathscr{Y})^2$,

(iv) $1 \leq 2\mathscr{H}(\Phi)\{\prod_v H_v((1_N - P_v)\boldsymbol{\xi})\}^2$,

where $P_v = P_v(\mathscr{X}_v)$ is projection onto the completion $\mathscr{X}_v$ of $\mathscr{X}$ in $(k_v)^N$.

The proof of this result is similar to our proof of [14, Theorem 1]. In fact we have made some technical simplifications which lead to the sharper inequality (1.6).

Theorem 5. *There exist $M - L$ linearly independent vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{M-L}$ in $\mathscr{Z} \backslash \mathscr{X}^{\perp}$ such that*

$$\mathscr{Z} = \operatorname{span}_k\{\mathscr{X}, \mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{M-L}\}, \tag{3.1}$$

and each of the subspaces

$$\mathscr{Y}_l = \operatorname{span}_k\{\mathscr{X}, \mathbf{z}_l\}, \qquad l = 1, 2, \ldots, M - L, \tag{3.2}$$

satisfies

$$H(\mathscr{Y}_l) \le 2c_k(1)\{c_k(M-L)\}^{M-L}\{2\mathscr{H}(\Phi)\}^{(M-L-1)/2}H(\mathscr{Z}). \tag{3.3}$$

In order to prove Theorem 1 from these results we set $\mathscr{X} = \mathscr{X}_0$. Then with $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_{M-L}$ as in Theorem 5 we apply Theorem 4 with $\boldsymbol{\xi} = \mathbf{z}_l$. It follows that the subspace $\mathscr{Y}_l = \operatorname{span}\{\mathscr{X}_0, \mathbf{z}_l\}$ contains an L-dimensional subspace $\mathscr{X}_l = \mathscr{X}_l'$ such that $\mathscr{X}_l \subseteq \mathscr{Z}^{(0)}$, $\dim(\mathscr{X}_0 \cap \mathscr{X}_l) = L-1$, and

$$\operatorname{span}\{\mathscr{X}_0, \mathscr{X}_1, \dots, \mathscr{X}_{M-L}\} = \mathscr{Z}.$$

Using (iii) of Theorem 4 we have

$$H(\mathscr{X}_0)^2 \le H(\mathscr{X}_0)H(\mathscr{X}_l) \le 2\mathscr{H}(\Phi)H(\mathscr{Y}_l)^2. \tag{3.4}$$

Then (3.3) and (3.4) combine to give exactly the estimate (1.2) in the statement of Theorem 1.

4. Proof of Theorem 4

Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_L\}$ be a basis for $\mathscr{X}$. Since $\boldsymbol{\xi} \in \mathscr{Z}\backslash\mathscr{X}^\perp$ it follows that

$$F(\mathbf{x}_l, \boldsymbol{\xi}) \ne 0 \quad \text{for some } l,\ 1 \le l \le L. \tag{4.1}$$

Obviously $\{\mathbf{x}_1, \dots, \mathbf{x}_L, \boldsymbol{\xi}\}$ is a basis for $\mathscr{Y}$. Define $\mathscr{X}' \subseteq \mathscr{Y}$ by

$$\mathscr{X}' = \left\{\sum_{l=1}^{L} \alpha_l \mathbf{x}_l + \beta\boldsymbol{\xi} : \alpha_l \in k, \beta \in k, \text{ and } \sum_{l=1}^{L} \alpha_l F(\mathbf{x}_l, \boldsymbol{\xi}) + \frac{1}{2}\beta F(\boldsymbol{\xi}) = 0\right\}.$$

In view of (4.1), $\mathscr{X}'$ is a subspace of $\mathscr{Y}$ with

$$\dim(\mathscr{X}') = L \quad \text{and} \quad \dim(\mathscr{X} \cap \mathscr{X}') = L-1.$$

In particular, $\mathscr{X}$ and $\mathscr{X}'$ are distinct subspaces. If

$$\mathbf{y} = \mathbf{x} + \beta\boldsymbol{\xi}, \qquad \mathbf{x} \in \mathscr{X},\ \beta \in k, \tag{4.2}$$

is a vector in $\mathscr{Y}$ then

$$\begin{aligned} F(\mathbf{y}) &= 2F(\mathbf{x}, \beta\boldsymbol{\xi}) + F(\beta\boldsymbol{\xi}) \\ &= 2\beta F(\mathbf{x} + \tfrac{1}{2}\beta\boldsymbol{\xi}, \boldsymbol{\xi}). \end{aligned} \tag{4.3}$$

This shows that $\beta = 0$ is equivalent to $\mathbf{y} \in \mathscr{X}$ and $F(\mathbf{x} + \frac{1}{2}\beta\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$ is equivalent to $\mathbf{y} \in \mathscr{X}'$. Therefore (i) and (ii) of Theorem 4 hold for the subspace $\mathscr{X}'$.

At each place v of k let $\mathscr{X}_v \subseteq (k_v)^N$ be the completion of $\mathscr{X}$ and let $\mathscr{X}_v' \subseteq (k_v)^N$ be the completion of $\mathscr{X}'$. Then set

$$P_v = P_v(\mathscr{X}_v), \qquad P_v' = P_v(\mathscr{X}_v'),$$

and

$$Q_v = \tfrac{1}{2}(1_N - P_v).$$

By continuity F must vanish identically on $\mathscr{X}_v \cup \mathscr{X}_v'$. With $\mathbf{y}$ given by (4.2) we have

$$
\begin{aligned}
F(\mathbf{x}+\tfrac{1}{2}\beta\boldsymbol{\xi},\boldsymbol{\xi}) &= F(\mathbf{x},\boldsymbol{\xi}) + \tfrac{1}{2}\beta F(\boldsymbol{\xi}) \\
&= F(\mathbf{x},(1_N - P_v)\boldsymbol{\xi}) + \tfrac{1}{2}\beta F((1_N + P_v)\boldsymbol{\xi},(1_N - P_v)\boldsymbol{\xi}) \\
&= F(Q_v\mathbf{x},(1_N - P_v)\boldsymbol{\xi}) + F(\beta Q_v\boldsymbol{\xi},(1_N - P_v)\boldsymbol{\xi}) \\
&= F(Q_v\mathbf{y},(1_N - P_v)\boldsymbol{\xi}).
\end{aligned}
\tag{4.4}
$$

If we combine (4.3) and (4.4) we find that

$$F(\mathbf{y}) = 2\beta F(Q_v\mathbf{y},(1_N - P_v)\boldsymbol{\xi}) \tag{4.5}$$

at each place v of k. It follows that $\mathbf{y}\in\mathscr{X}'$ if and only if

$$F(Q_v\mathbf{y},(1_N - P_v)\boldsymbol{\xi}) = 0.$$

Now select $\mathbf{y}\in\mathscr{Y}\backslash(\mathscr{X}\cup\mathscr{X}')$. Then $P_v'\mathbf{y}\in\mathscr{X}_v'$ and so at each place v

$$F(Q_vP_v'\mathbf{y},(1_N - P_v)\boldsymbol{\xi}) = 0. \tag{4.6}$$

Combining (4.5) and (4.6) we obtain

$$0 \neq F(\mathbf{y}) = 2\beta F(Q_v(1_N - P_v')\mathbf{y},(1_N - P_v)\boldsymbol{\xi}). \tag{4.7}$$

Next we apply the product formula to (4.7) and use the basic inequality (2.6) at each place. This leads to

$$
\begin{aligned}
1 &= \prod_v |(2\beta)^{-1}F(\mathbf{y})|_v \\
&\leq \mathscr{H}(\Phi)\left\{\prod_v H_v(Q_v(1_N - P_v')\mathbf{y})\right\}\left\{\prod_v H_v((1_N - P_v)\boldsymbol{\xi})\right\}.
\end{aligned}
\tag{4.8}
$$

By [14, Lemma 8] we may remove the operator Q_v from the right-hand side of (4.8) while compensating with an extra factor of 2. In this way we find that

$$1 \leq 2\mathscr{H}(\Phi)\left\{\prod_v H_v((1_N - P_v')\mathbf{y})\right\}\left\{\prod_v H_v((1_N - P_v)\boldsymbol{\xi})\right\}. \tag{4.9}$$

Finally we multiply both sides of (4.9) by $H(\mathscr{X})H(\mathscr{X}')$ and apply (2.1). This establishes the inequality

$$
\begin{aligned}
H(\mathscr{X})H(\mathscr{X}') &\leq 2\mathscr{H}(\Phi)H(\mathrm{span}_k\{\mathscr{X}',\mathbf{y}\})H(\mathrm{span}_k\{\mathscr{X},\boldsymbol{\xi}\}) \\
&= 2\mathscr{H}(\Phi)H(\mathscr{Y})^2,
\end{aligned}
\tag{4.10}
$$

which is exactly the second inequality in (iii) of the theorem.

Now $\mathscr{X}\subseteq\mathscr{Z}^{(0)}$ has minimal height among all L-dimensional subspaces contained in $\mathscr{Z}^{(0)}$. Therefore (4.10) immediately implies that

$$
\begin{aligned}
H(\mathscr{X})^2 &\leq H(\mathscr{X})H(\mathscr{X}') \leq 2\mathscr{H}(\Phi)H(\mathscr{Y})^2 \\
&= 2\mathscr{H}(\Phi)H(\mathscr{X})^2\left\{\prod_v H_v((1_N - P_v)\boldsymbol{\xi})\right\}^2.
\end{aligned}
\tag{4.11}
$$

Of course (4.11) is (iv) and the first inequality in (iii) of the theorem.

5. A NESTED SEQUENCE OF SUBSPACES

In this section we identify several objects connected with F, $\mathscr{X}$ and $\mathscr{Z}$ which will be used in our proof of Theorem 5. Again we suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_L\}$ is a basis for $\mathscr{X}$ and we write $P_v = P_v(\mathscr{X}_v)$ for the projection operator which maps $(k_v)^N$ onto the completion $\mathscr{X}_v$ of $\mathscr{X}$ at each place v. To simplify some expressions we write

$$G(\mathbf{z}) = \prod_v H_v((1_N - P_v)\mathbf{z})$$

if $\mathbf{z} \in \mathscr{Z}$. By [14, Theorem 10] there exists a basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_L, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{M-L}\}$ of $\mathscr{Z}$ such that

$$\prod_{l=1}^{M-L} G(\mathbf{y}_l) \le \{c_k(M-L)\}^{M-L} H(\mathscr{Z}) H(\mathscr{X})^{-1}. \tag{5.1}$$

Also, by reordering if necessary, we may assume that the numbers $G(\mathbf{y}_l)$ are increasing for $l = 1, 2, \ldots, M-L$. We use the vectors in $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{M-L}\}$ to define a nested sequence of subspaces

$$\mathscr{A}_l = \operatorname{span}_k\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_L, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_l\},$$

for $l = 0, 1, 2, \ldots, M-L$. Thus we have

$$\mathscr{X} = \mathscr{A}_0 \subseteq \mathscr{A}_1 \subseteq \mathscr{A}_2 \subseteq \cdots \subseteq \mathscr{A}_{M-L} = \mathscr{Z}.$$

Obviously $\dim(\mathscr{A}_l) = L + l$. For each choice of $l = 0, 1, 2, \ldots, M-L$ we set

$$\mathscr{A}_l^{\perp} = \{\mathbf{z} \in \mathscr{Z} : F(\mathbf{a}, \mathbf{z}) = 0 \text{ for all } \mathbf{a} \in \mathscr{A}_l\}.$$

Clearly each $\mathscr{A}_l^{\perp}$ is a subspace of $\mathscr{Z}$ and

$$\mathscr{Z}^{\perp} = \mathscr{A}_{M-L}^{\perp} \subseteq \mathscr{A}_{M-L-1}^{\perp} \subseteq \cdots \subseteq \mathscr{A}_1^{\perp} \subseteq \mathscr{A}_0^{\perp} = \mathscr{X}^{\perp} \subseteq \mathscr{Z}.$$

Concerning the dimension of $\mathscr{A}_l^{\perp}$ we have

$$\dim(\mathscr{A}_l) + \dim(\mathscr{A}_l^{\perp}) = \dim(\mathscr{Z}) + \dim(\mathscr{A}_l \cap \mathscr{Z}^{\perp}).$$

Since $r > M - L$ it follows that

$$\begin{aligned} \dim(\mathscr{A}_l^{\perp}) &\le \dim(\mathscr{Z}) + \dim(\mathscr{Z}^{\perp}) - \dim(\mathscr{A}_l) \\ &= M + (M - r) - (L + l) \\ &\le M - l - 1. \end{aligned} \tag{5.2}$$

The quadratic form F vanishes identically on $\mathscr{X} = \mathscr{A}_0$ but does not vanish identically on $\mathscr{Z} = \mathscr{A}_{M-L}$. Hence there exists a unique integer s, $0 \le s \le M - L - 1$, such that F vanishes identically on $\mathscr{A}_s$ but F does not vanish identically on $\mathscr{A}_{s+1}$. For the subspace $\mathscr{A}_s$ we must have $\mathscr{A}_s \subseteq \mathscr{A}_s^{\perp}$, and therefore by (5.2)

$$L + s = \dim(\mathscr{A}_s) \le \dim(\mathscr{A}_s^{\perp}) \le M - s - 1.$$

This shows that $0 \le s \le \frac{1}{2}(M - L - 1)$.

Lemma 6. *If* $\mathbf{y}_{s+1} \in \mathscr{A}_s^{\perp}$ *then*

$$1 \leq \mathscr{H}(\Phi) G(\mathbf{y}_{s+1})^2.$$

Proof. By the definition of s, F does not vanish identically on

$$\mathscr{A}_{s+1} = \operatorname{span}_k\{\mathscr{A}_s, \mathbf{y}_{s+1}\}.$$

But

$$F(\mathbf{a}) = 0 \quad \text{and} \quad F(\mathbf{a}, \mathbf{y}_{s+1}) = 0 \tag{5.3}$$

for all $\mathbf{a} \in \mathscr{A}_s$. It follows that $F(\mathbf{y}_{s+1}) \neq 0$. Also, (5.3) implies that at each place v

$$F(P_v \mathbf{y}_{s+1}) = 0 \quad \text{and} \quad F(P_v \mathbf{y}_{s+1}, \mathbf{y}_{s+1}) = 0.$$

Therefore, we have

$$\begin{aligned} F(\mathbf{y}_{s+1}) &= F((1_N - P_v)\mathbf{y}_{s+1}, \mathbf{y}_{s+1}) \\ &= F((1_N - P_v)\mathbf{y}_{s+1}). \end{aligned}$$

We apply (2.6) and the product formula to conclude that

$$\begin{aligned} 1 &= \prod_v |F(\mathbf{y}_{s+1})|_v \\ &= \prod_v |F((1_N - P_v)\mathbf{y}_{s+1})|_v \\ &\leq \mathscr{H}(\Phi) G(\mathbf{y}_{s+1})^2. \end{aligned}$$

Lemma 7. *Suppose that* $1 \leq s$ *and* $\boldsymbol{\xi}$ *is a vector in* $\mathscr{Z} \backslash \mathscr{A}_s^{\perp}$. *Let* m *be the smallest integer in the set* $\{1, 2, \ldots, s\}$ *such that* $\boldsymbol{\xi}$ *is contained in* $\mathscr{Z} \backslash \mathscr{A}_m^{\perp}$. *Then we have*

$$1 \leq \mathscr{H}(\Phi) G(\mathbf{y}_m) G(\boldsymbol{\xi}). \tag{5.4}$$

Proof. First we assume that $m = 1$. Then $\boldsymbol{\xi} \in \mathscr{Z} \backslash \mathscr{A}_1^{\perp}$ and therefore the linear form $\mathbf{a} \to F(\mathbf{a}, \boldsymbol{\xi})$ is not trivial on $\mathscr{A}_1$. It follows that

$$\mathscr{B}_0 = \{\mathbf{a} \in \mathscr{A}_1 : F(\mathbf{a}, \boldsymbol{\xi}) = 0\}$$

is an L-dimensional subspace of $\mathscr{A}_1$. As $1 \leq s$, F vanishes identically on $\mathscr{A}_1$ and also on $\mathscr{B}_0$. Since $\mathscr{X} = \mathscr{A}_0$ has minimal height among the L-dimensional subspaces in $\mathscr{Z}^{(0)}$, we have

$$H(\mathscr{A}_0) \leq H(\mathscr{B}_0). \tag{5.5}$$

Now let $\boldsymbol{\zeta} \in \mathscr{A}_1 \backslash \mathscr{B}_0$ so that $F(\boldsymbol{\zeta}, \boldsymbol{\xi}) \neq 0$. From the definition of $\mathscr{B}_0$ we have

$$F(P_v(\mathscr{B}_0)\boldsymbol{\zeta}, \boldsymbol{\xi}) = 0 \tag{5.6}$$

at each place v. Since F vanishes identically on $\mathscr{A}_1$ we also have

$$F((1_N - P_v(\mathscr{B}_0))\boldsymbol{\zeta}, P_v(\mathscr{A}_0)\boldsymbol{\xi}) = 0. \tag{5.7}$$

Using (5.6) and (5.7) we obtain the identity

$$\begin{aligned} F(\boldsymbol{\zeta},\boldsymbol{\xi}) &= F((1_N - P_v(\mathscr{B}_0))\boldsymbol{\zeta},\boldsymbol{\xi}) \\ &= F((1_N - P_v(\mathscr{B}_0))\boldsymbol{\zeta},(1_N - P_v(\mathscr{A}_0))\boldsymbol{\xi}) \end{aligned}$$

at each place v. Again we apply (2.6) and the product formula to conclude that

$$\begin{aligned} 1 &= \prod_v |F(\boldsymbol{\zeta},\boldsymbol{\xi})|_v \\ &\leq \mathscr{H}(\Phi)\left\{\prod_v H_v((1_N - P_v(\mathscr{B}_0))\boldsymbol{\zeta})\right\}\left\{\prod_v H_v((1_N - P_v(\mathscr{A}_0))\boldsymbol{\xi})\right\}. \end{aligned} \tag{5.8}$$

Next we multiply both sides of (5.8) by $H(\mathscr{B}_0)$ and use (2.1) and (5.5). In this way we obtain

$$\begin{aligned} H(\mathscr{A}_0) \leq H(\mathscr{B}_0) &\leq \mathscr{H}(\Phi)H(\mathscr{A}_1)\left\{\prod_v H_v((1_N - P_v(\mathscr{A}_0))\boldsymbol{\xi})\right\} \\ &= \mathscr{H}(\Phi)H(\mathscr{A}_0)\left\{\prod_v H_v((1_N - P_v(\mathscr{A}_0))\mathbf{y}_1)\right\}\left\{\prod_v H_v((1_N - P_v(\mathscr{A}_0))\boldsymbol{\xi})\right\}, \end{aligned}$$

which is (5.4) when $m = 1$.

To complete the proof we consider the case $2 \leq m \leq s$. Then $\boldsymbol{\xi} \in \mathscr{Z}\backslash\mathscr{A}_m^{\perp}$ and $\boldsymbol{\xi} \in \mathscr{A}_{m-1}^{\perp}$. Again the linear form $\mathbf{a} \to F(\mathbf{a},\boldsymbol{\xi})$ is not trivial on $\mathscr{A}_m$ so that

$$\{\mathbf{a} \in \mathscr{A}_m : F(\mathbf{a},\boldsymbol{\xi}) = 0\}$$

is an $(L+m-1)$-dimensional subspace of $\mathscr{A}_m$. On the other hand, $\mathscr{A}_{m-1} \subseteq \mathscr{A}_m$, $\boldsymbol{\xi} \in \mathscr{A}_{m-1}^{\perp}$ and therefore $\mathbf{a} \to F(\mathbf{a},\boldsymbol{\xi})$ is identically zero on $\mathscr{A}_{m-1}$. In other words,

$$\mathscr{A}_{m-1} = \{\mathbf{a} \in \mathscr{A}_m : F(\mathbf{a},\boldsymbol{\xi}) = 0\}.$$

The vector $\mathbf{y}_m$ is in $\mathscr{A}_m\backslash\mathscr{A}_{m-1}$ and therefore $F(\mathbf{y}_m,\boldsymbol{\xi}) \neq 0$. As in the first part of the proof we have

$$\begin{aligned} F(\mathbf{y}_m,\boldsymbol{\xi}) &= F((1_N - P_v(\mathscr{A}_0))\mathbf{y}_m,\boldsymbol{\xi}) \\ &= F((1_N - P_v(\mathscr{A}_0))\mathbf{y}_m,(1_N - P_v(\mathscr{A}_0))\boldsymbol{\xi}). \end{aligned} \tag{5.9}$$

Using (2.1), (5.9) and the product formula we find that

$$1 = \prod_v |F(\mathbf{y}_m,\boldsymbol{\xi})|_v \leq \mathscr{H}(\Phi)G(\mathbf{y}_m)G(\boldsymbol{\xi}).$$

This proves the lemma.

Lemma 8. *The vectors* $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{M-L-1}$ *satisfy*

$$1 \leq \{2\mathscr{H}(\Phi)\}^{(M-L-1)/2}\prod_{l=1}^{M-L-1} G(\mathbf{y}_l). \tag{5.10}$$

The vector $\mathbf{y}_{M-L}$ *satisfies*

$$G(\mathbf{y}_{M-L}) \le \{c_k(M-L)\}^{M-L}\{2\mathscr{H}(\Phi)\}^{(M-L-1)/2}H(\mathscr{Z})H(\mathscr{X})^{-1}. \tag{5.11}$$

Proof. The inequalities (5.1) and (5.10) clearly imply that (5.11) holds. Therefore it suffices to establish only (5.10). First we consider the case $s = 0$. If $\mathbf{y}_1 \in \mathscr{A}_0^{\perp}$ then

$$1 \le \mathscr{H}(\Phi)G(\mathbf{y}_1)^2$$

holds by Lemma 6. If $\mathbf{y}_1 \in \mathscr{Z}\backslash\mathscr{A}_0^{\perp}$ we may appeal to Theorem 4 part (iv) with $\boldsymbol{\xi} = \mathbf{y}_1$. We find that

$$1 \le 2\mathscr{H}(\Phi)G(\mathbf{y}_1)^2. \tag{5.12}$$

Thus (5.12) holds generally if $s = 0$. Since the numbers $G(\mathbf{y}_l)$ are increasing for $l = 1, 2, \dots, M-L$, the desired inequality (5.10) follows immediately.

For the remainder of the proof we assume that $1 \le s$. Let $0 \le l \le s$ so that

$$\mathscr{A}_l \subseteq \mathscr{A}_s \subseteq \mathscr{A}_s^{\perp} \subseteq \mathscr{A}_l^{\perp}.$$

In particular,

$$\mathscr{A}_s = \operatorname{span}_k\{\mathscr{A}_0, \mathbf{y}_1, \dots, \mathbf{y}_s\} \subseteq \mathscr{A}_l^{\perp}$$

and $\dim(\mathscr{A}_l^{\perp}) \le M - l - 1$ by (5.2). It follows that at most $M-L-s-l-1$ of the vectors in the set $\{\mathbf{y}_{s+1}, \mathbf{y}_{s+2}, \dots, \mathbf{y}_{M-L}\}$ are also in $\mathscr{A}_l^{\perp}$. Alternatively, at least $l+1$ vectors in the set $\{\mathbf{y}_{s+1}, \mathbf{y}_{s+2}, \dots, \mathbf{y}_{M-L}\}$ are also in $\mathscr{Z}\backslash\mathscr{A}_l^{\perp}$. Hence we may select distinct integers $i_0, i_1, \dots, i_s$ in $\{s+1, s+2, \dots, M-L\}$ so that

$$\{\mathbf{y}_{i_0}, \mathbf{y}_{i_1}, \dots, \mathbf{y}_{i_l}\} \subseteq \mathscr{Z}\backslash\mathscr{A}_l^{\perp}$$

for each $l = 0, 1, 2, \dots, s$. By removing $\mathbf{y}_{M-L}$ if necessary, we may select distinct integers $j_1, j_2, \dots, j_s$ in the set $\{s+1, s+2, \dots, M-L-1\}$ so that

$$\{\mathbf{y}_{j_1}, \mathbf{y}_{j_2}, \dots, \mathbf{y}_{j_l}\} \subseteq \mathscr{Z}\backslash\mathscr{A}_l^{\perp}$$

for $l = 1, 2, \dots, s$. Since $\mathscr{Z}\backslash\mathscr{A}_l^{\perp} \subseteq \mathscr{Z}\backslash\mathscr{A}_s^{\perp}$ the hypotheses of Lemma 7 hold with $\boldsymbol{\xi} = \mathbf{y}_{j_l}$. The corresponding value of m plainly satisfies $1 \le m \le l$. Thus by Lemma 7 we have

$$1 \le \mathscr{H}(\Phi)G(\mathbf{y}_m)G(\mathbf{y}_{j_l}) \le \mathscr{H}(\Phi)G(\mathbf{y}_l)G(\mathbf{y}_{j_l}), \tag{5.13}$$

for each $l = 1, 2, \dots, s$.

Next we claim that

$$1 \le \mathscr{H}(\Phi)G(\mathbf{y}_{s+1})^2. \tag{5.14}$$

If $\mathbf{y}_{s+1} \in \mathscr{A}_s^{\perp}$ then this follows from Lemma 6. If $\mathbf{y}_{s+1} \in \mathscr{Z}\backslash\mathscr{A}_s^{\perp}$ we may apply Lemma 7 with $\boldsymbol{\xi} = \mathbf{y}_{s+1}$. Then we use the trivial inequality $G(\mathbf{y}_m) \le G(\mathbf{y}_{s+1})$ (where $1 \le m \le s$) to deduce that (5.14) holds in this case as well.

Now suppose that t is an integer, $s+1 \leq t \leq M-L-1$, but t is *not* in the set $\{j_1, j_2, \ldots, j_s\}$. Obviously $G(\mathbf{y}_{s+1}) \leq G(\mathbf{y}_t)$ and therefore (5.14) implies that

$$1 \leq \mathscr{H}(\Phi)G(\mathbf{y}_t)^2. \tag{5.15}$$

There are $M-L-2s-1$ such integers t so that (5.15) leads to the inequality

$$1 \leq \mathscr{H}(\Phi)^{(M-L-1)/2-s}\prod_t G(\mathbf{y}_t). \tag{5.16}$$

From (5.13) we have

$$1 \leq \mathscr{H}(\Phi)^s \left\{\prod_{l=1}^{s} G(\mathbf{y}_l)G(\mathbf{y}_{j_l})\right\}. \tag{5.17}$$

Finally, the inequalities (5.16) and (5.17) combine to establish (5.10).

6. Proof of Theorem 5

We have already seen that there exists an integer i_0, $s+1 \leq i_0 \leq M-L$, such that $\mathbf{y}_{i_0}$ is contained in $\mathscr{Z}\backslash\mathscr{A}_0^{\perp} = \mathscr{Z}\backslash\mathscr{X}^{\perp}$. Using $\mathbf{y}_{i_0}$ and the linearly independent vectors $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{M-L}\}$ we define a second set of linearly independent vectors $\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{M-L}\}$ as follows.

(i) If $\mathbf{y}_l \in \mathscr{Z}\backslash\mathscr{X}^{\perp}$ we set $\mathbf{z}_l = \mathbf{y}_l$.

(ii) If $\mathbf{y}_l \in \mathscr{X}^{\perp}$ we select a scalar $\alpha_l \neq 0$ in k such that

$$G(\alpha_l\mathbf{y}_l + \mathbf{y}_{i_0}) \leq 2c_k(1)\max\{G(\mathbf{y}_l), G(\mathbf{y}_{i_0})\}.$$

That such a scalar $\alpha_l \neq 0$ exists follows from Lemma 3. Then we set $\mathbf{z}_l = \alpha_l\mathbf{y}_l + \mathbf{y}_{i_0}$.

It is clear that

$$\operatorname*{span}_k\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{M-L}\} = \operatorname*{span}_k\{\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{M-L}\}$$

and therefore (3.1) holds. Also, each vector $\mathbf{z}_l$ is in $\mathscr{Z}\backslash\mathscr{X}^{\perp}$.

To complete the proof we note that

$$G(\mathbf{z}_l) \leq 2c_k(1)G(\mathbf{y}_{M-L}) \tag{6.1}$$

for each l, $1 \leq l \leq M-L$. If $\mathscr{Y}_l$ is defined by (3.2) then

$$H(\mathscr{Y}_l) = H(\mathscr{X})G(\mathbf{z}_l). \tag{6.2}$$

The inequality (3.3) plainly follows from (5.11), (6.1) and (6.2).

7. Proof of Theorem 2

Since the rank of F restricted to $\mathscr{Z}$ is r there exists a subspace $\mathscr{Y} \subseteq \mathscr{Z}$ such that $\mathscr{Y}$ has dimension r and F restricted to $\mathscr{Y}$ is nonsingular. By Theorem 10 of [14] there exists a basis $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_r\}$ of $\mathscr{Y}$ for which

$$\prod_{l=1}^{r}\left\{\prod_v H_v\{(1_N - P_v(\mathscr{Z}^{\perp}))\mathbf{y}_l\}\right\} \leq c_k(r)^r H(\mathscr{Z})H(\mathscr{Z}^{\perp})^{-1}. \tag{7.1}$$

Let

$$Y = (\mathbf{y}_1 \mathbf{y}_2 \cdots \mathbf{y}_r)$$

denote the $N \times r$ matrix having $\mathbf{y}_l$ as its lth column, $1 \le l \le r$. We must have

$$\det\{Y^T \Phi Y\} \neq 0,$$

and therefore when we expand the determinant

$$\sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{l=1}^{r} F(\mathbf{y}_l, \mathbf{y}_{\sigma(l)}) \neq 0. \tag{7.2}$$

In (7.2) the sum is over all permutations σ of the set $\{1, 2, \ldots, r\}$. It follows that there exists a permutation τ of the set $\{1, 2, \ldots, r\}$ such that

$$\prod_{l=1}^{r} F(\mathbf{y}_l, \mathbf{y}_{\tau(l)}) \neq 0.$$

Using (2.1) and the product formula we find that

$$\begin{aligned} 1 &= \prod_{l=1}^{r} \left\{ \prod_{v} |F(\mathbf{y}_l, \mathbf{y}_{\tau(l)})|_v \right\} \\ &= \prod_{l=1}^{r} \left\{ \prod_{v} |F((1_N - P_v(\mathscr{Z}^{\perp}))\mathbf{y}_l, (1_N - P_v(\mathscr{Z}^{\perp}))\mathbf{y}_{\tau(l)})|_v \right\} \\ &\le \prod_{l=1}^{r} \{\mathscr{H}(\Phi) G(\mathbf{y}_l) G(\mathbf{y}_{\tau(l)})\} \\ &= \mathscr{H}(\Phi)^r \prod_{l=1}^{r} G(\mathbf{y}_l)^2, \end{aligned} \tag{7.3}$$

where we have written

$$G(\mathbf{x}) = \prod_{v} H_v\{(1_N - P_v(\mathscr{Z}^{\perp}))\mathbf{x}\}.$$

Finally, from (7.1) and (7.3) we obtain the inequality

$$1 \le \mathscr{H}(\Phi)^r \{c_k(r)^r H(\mathscr{Z}) H(\mathscr{Z}^{\perp})^{-1}\}^2,$$

which is the statement of the theorem.

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